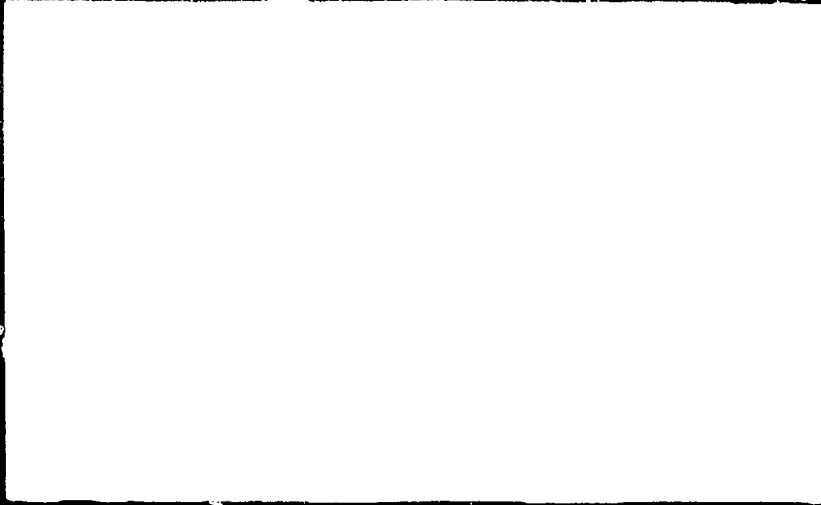


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K-CONNECTIVITY IN RANDOM UNDIRECTED GRAPHS

BY

John H. Reif and Paul G. Spirakis

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This paper concerns vertex connectivity in random graphs. We present results bounding the cardinality of the biggest k -block in random graphs of the $G_{n,p}$ model, for any constant value of k . These results generalize those of [Erdős, Renyi, 60] and [Karp, Tarjan, 80] for $k=1$ and 2. We furthermore prove here that the cardinality of the biggest k -block is $\approx n \log n$ with probability $\approx 1 - n^{-2}$ for $p \geq c_1^*(k)/n$ and $c_1^*(k) > k+2$. We also show that if $p \geq c(k) \frac{(\log n)^{1/k}}{n}$ with $c(k) > 32k^2$ then the graph $G_{n,p}$ is k -connected with probability $\approx 1 - 2n^{-d'(k)}$, $d'(k) > 1$.

K-CONNECTIVITY IN RANDOM UNDIRECTED GRAPHS*

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1. Summary

This paper concerns vertex connectivity in random graphs. We present results bounding the cardinality of the biggest k -block in random graphs of the $G_{n,p}$ model, for any constant value of k . These results generalize those of [Erdős, Renyi, 60] and [Karp, Tarjan, 80] for $k=1$ and 2. We furthermore prove here that the cardinality of the biggest k -block is $\geq n - \log n$ with probability $\geq 1 - n^{-2}$ for $p \geq c_1(k)/n$ and $c_1(k) > k+2$. We also show that if $p \geq c(k) \frac{\log n}{n}$ with $c(k) > 32k^2$ then the graph $G_{n,p}$ is k -connected with probability $\geq 1 - 2n^{-d'(k)}$, $d'(k) > 1$.

2. Introduction

A graph $G = (V, E)$ consists of a finite nonempty set V of vertices together with a prescribed set E of unordered pairs of *distinct* elements of V (set of edges). (We allow no loops neither multiple edges). The *vertex connectivity* $k(G)$ of an undirected graph G is the minimum number of vertices whose removal results in a disconnected graph or a *trivial* graph (consisting of just one vertex). Note that we follow here [Matula, 78] in defining k -connectivity, which we find to be most natural. [McLane, 37] gives a (somewhat different) definition of triconnectivity so that he can have the theorem that a graph is planar if its triconnected components are. [McLane, 37] shows that his triconnected components are homeomorphic to 3-blocks. Vertex k -connectivity seems to be a fundamental property of a graph and has numerous applications to other graph problems (such as planarity testing, routing problems etc). It is relevant to questions concerning vulnerability of a graph to separation. Cluster analysis methods considering the nature and inherent reliability of proximity

data use the theory of k -connectivity to find groups of likes and dislikes in object pair association graphs ([Matula, 77], [Matula, 78] also [Jardine, Sibson, 71]).

A k -block of an undirected graph G is a maximal k -connected subgraph. A k -block is *trivial* if it has only one vertex [Matula, 78]. Clearly, each k -block consists of $\geq k$ vertices or it is trivial.

[Matula, 78] examined certain properties of k -blocks in graphs (number of them, separation lemma) and [Erdős, Renyi, 60] and [Karp, Tarjan, 80] examined the distribution of the size of the biggest 1 and 2-blocks in random graphs $G_{n,p}$ with $p \geq \frac{c}{n}$ and $G_{n,N}$ with $N \geq cn$. They proved that there is a giant k -block for $k=1,2$, with exponentially decaying probability of error. For $p > \frac{1}{2} \frac{\log n}{n}$ [Erdős, Renyi 60] showed that $G_{n,p}$ becomes almost surely 2-connected.

In our paper we examine k -connectivity in the model $G_{n,p}$, defined precisely as follows: For $0 \leq p \leq 1$ and $n \geq 0$ let $G_{n,p}$ be a random variable whose values are graphs on the vertex set $\{1, 2, \dots, n\}$. If $e = \{u, v\}$ and $u, v \in \{1, 2, \dots, n\}$ then $\text{Prob}\{e \text{ is an edge}\} = p$ and these probabilities are independent for different e .

We prove that for each constant $k \geq 0$ and for each ε ($0 < \varepsilon \leq 1$) and $\alpha > 1$, there is a k -block of cardinality $\geq \varepsilon n$ in $G_{n,p}$ with $p \geq \frac{c(k, \varepsilon, \alpha)}{n}$ with probability $\geq 1 - e^{-\alpha n}$. We furthermore prove that for any $k > 0$ and $0 \leq m < \frac{n}{2k}$ there are constants $c(k), d(k) > 0$ such that the size of the biggest k -block of $G_{n,p}$ where $p \geq c(k) \frac{\log n}{n}$ is equal to $n-m$ with probability $n^{-m \cdot d(k)}$. From that we get as corollaries, that there are $c(k), d(k) > 0$ and $d'(k) > 1$ such that the size of the biggest k -block of $G_{n,p}$ is $\geq n - \log n$ with $\text{prob} > 1 - 2n^{-d'(k) \log n}$ and that $G_{n,p}$ is k -connected with $\text{prob} > 1 - 2n^{-d'(k)}$.

Finally, we prove that for any $m = o(n)$ $\exists c_1(k) > k+2$ and a function $t(n) \geq \frac{c_1(k) \log n}{m}$ such that, if $p \geq \frac{t(n)}{n}$ then the biggest k -block of $G_{n,p}$ has size $> n-m$ with probability $> 1 - n^k/e^{t(n)m} \rightarrow 1$ as $n \rightarrow \infty$. A corollary is that if $p \geq \frac{c_1(k)}{n}$ then the biggest k -block of $G_{n,p}$ has cardinality $> n - \log n$ with probability $\geq 1 - n^{-2}$. These results were known by [Erdős, Renyi, 60] only for $k=1$ and $c(1) \geq \frac{1}{2}$.

3. Properties of k -blocks

PROPOSITION 1 [Matula, 78] For each $k \geq 0$, any two k -blocks have no more than $k-1$ vertices in common.

DEFINITION [Matula, 78] A *separation set* S of G is a vertex subset $S \subseteq V(G)$ such that $G - S$ is disconnected. A *minimum separating set* $S \subseteq V(G)$ has $|S| = k(G)$.

DEFINITION Let G be a graph (V, E) and let $S \subseteq V$ be a set of vertices. Then by $\langle S \rangle$ we denote the subgraph induced by S on G .

LEMMA 1 [Matula, 78] (Block separation lemma) Let $S \subseteq V(G)$ be a minimum separating set of the noncomplete graph G with $\langle A_1 \rangle, \langle A_2 \rangle, \dots, \langle A_m \rangle$, $m \geq 2$ the components of $G - \langle S \rangle$ and let $k \geq k(G) + 1$. Then each k -block of G is a k -block of $\langle A_i \cup S \rangle$ for precisely one value of i , and each k -block of $\langle A_i \cup S \rangle$ for every i is a k -block of G .

For a proof, see [Matula, 78].

REMARK [Matula, 78] shows that for each $k \geq 1$ the total number of nontrivial k' -blocks for $1 \leq k' \leq k$, is $\leq \left\lfloor \frac{2n-1}{3} \right\rfloor$ for any graph G with n vertices.

4. Giant k-blocks in Random Graphs

In the following we introduce special notation for very large subgraphs. For each ϵ , $0 \leq \epsilon \leq 1$, a subgraph H of a graph G of n vertices is called an ϵ -giant of G if the cardinality of the vertex set of H is $\geq \epsilon n$.

DEFINITION: Given a vertex set $S \subseteq V$ in the graph $G = (V, E)$, the boundary vertices of S is the set $B(S) = \{u \in S \mid \exists v \in V-S \text{ such that } \{u, v\} \in E\}$.

DEFINITION: Let X be a random variable whose values are the cardinality of the maximum k -block of instances of $G_{n,p}$. Let $F_{n,p,k}(a) = \text{Prob}\{X \leq a\}$ be the distribution function of X .

THEOREM 1: For every ϵ on $(0,1)$, $\alpha > 1$ and $k > 0$ there is a $c = c(k, \epsilon, \alpha) > 0$ such that, for $p \geq \frac{c}{n}$, $F_{n,p,k}(\epsilon n) \leq e^{-\alpha n}$. In other words, the random graph $G_{n,p}$ with $p \geq \frac{c}{n}$ has an ϵ -giant k -block with probability at least $1 - e^{-\alpha n}$. To prove this theorem, we shall need the following definition and lemma.

DEFINITION: If $G = (V, E)$ and A, B are subsets of V , then $E(A, B) = \{e = \{u, v\} \in E \mid u \in A \text{ and } v \in B\}$.

LEMMA 2: For any $\alpha_1, \epsilon_1, \epsilon_2 > 0$ where $\epsilon_1 + \epsilon_2 \leq 1$ and $\alpha_1 \geq 1$ there are constants $c, \epsilon_3, \epsilon_4 > 0$ such that a random graph $G_{n,p}$ with $p \geq \frac{c}{n}$ has the property (*) with probability $\geq 1 - e^{-\alpha_1 n}$.

(*): If A, B are any two vertex subsets of V such that $|A| \geq \lfloor \epsilon_1 n \rfloor$, $|B| \geq \lfloor \epsilon_2 n \rfloor$ and $A \cap B = \emptyset$ then $|E(A, B)| > 0$.

PROOF OF LEMMA: The complement of (*) is: "There are two vertex subsets A, B such that $|A| \geq \lfloor \epsilon_1 n \rfloor$, $|B| \geq \lfloor \epsilon_2 n \rfloor$, $A \cap B = \emptyset$ and

$E(A,B) = \emptyset$. Clearly

$$\text{Prob}\{E(A,B) = \emptyset\} \leq (1-p)^{\epsilon_1 n \epsilon_2 n} \leq \left(1 - \frac{c}{n}\right)^{\epsilon_1 \epsilon_2 n} \leq e^{-c \epsilon_1 \epsilon_2 n}$$

Since there are at most $\frac{1}{2} \cdot 4^n$ ways to select these A, B , and upper bound on the probability of the complement of (*) is

$$\begin{aligned} \sum_{\text{all } A, B} \text{Prob}\{E(A,B) = \emptyset\} \\ \leq \frac{1}{2} \left(4e^{-c \epsilon_1 \epsilon_2}\right)^n \leq e^{-\alpha_1 n} \end{aligned}$$

for

$$c \geq \frac{\alpha_1 + \log_e 4}{\epsilon_1 \epsilon_2}$$

□

Now we return to the proof of the Theorem 1. Let $G = (V, E)$ be an instance of the random graph $G_{n,p}$. Let \mathcal{E}_1 be the event "G has no ϵ -giant k -block". Assume event \mathcal{E}_1 be true in the instance G of $G_{n,p}$. Let initially the set $A = \emptyset$. Do the following construction just until A has cardinality $\geq \epsilon' n/2$, where $\epsilon' = \min(\epsilon, 1-\epsilon)$.

(a) Find a minimum separating set S of G . Let $\langle A_1 \rangle, \dots, \langle A_m \rangle$ $m \geq 2$ be the components of $G-S$. Let $\langle A_i \rangle$ be the smallest of them. Let $A \leftarrow (A_i \cup S) \cup A$. Let B be the union of the rest of the components and let $G \leftarrow$ the graph induced by $B \cup S$. If $|A| < \epsilon' \cdot \frac{n}{2}$, then go to (a).

By the above method of constructing A , each addition of a component in A adds at most $k-1$ vertices to $B(A)$ (i.e. the vertices of the

cut) and at least one vertex to $A - B(A)$ (by the block separation lemma and by the fact that k -blocks have $\geq k$ vertices if they are non-trivial) or causes the transformation of a boundary to a nonboundary vertex. Thus, at least $1/k$ of the vertices of A are not in $B(A)$.

By this construction, finally the k -blocks of G are going to be separated. Because all k -blocks have been assumed to have cardinality $\leq \epsilon n$, we will finally have

$$\epsilon' \frac{n}{2} \leq |A| \leq \min \left[\epsilon' \frac{n}{2} + \epsilon n, \epsilon' \frac{n}{2} \frac{3}{2} \right]$$

So

$$|A - B(A)| \geq \frac{\min(\epsilon, 1-\epsilon)}{2k} \cdot n$$

and

$$|V - A| \geq n \left(1 - \min \left[\left(\epsilon + \epsilon'/2 \right), \left(3\epsilon'/4 \right) \right] \right)$$

(obviously $|V - A| > 0$ for any ϵ on $(0,1)$). Let $Y = A - B(A)$ and $Z = V - A$ then $|Y| \geq \epsilon_1 n$ and $|Z| \geq \epsilon_2 n$ where $\epsilon_1 = \frac{\epsilon'}{2k}$, $\epsilon_2 = 1 - \min \left[\left(\epsilon + \frac{1}{2}\epsilon' \right), \left(3\epsilon'/4 \right) \right]$ and $E(Y, Z) = \emptyset$ by construction

Hence, there are disjoint sets $Y' \subseteq Y$ and $Z' \subseteq Z$ such that $|Y'| = \epsilon_1 n$, $|Z'| = \epsilon_2 n$ and $E(Y', Z') = \emptyset$. Call \mathcal{E}_2 the above event. We have just shown \mathcal{E}_1 implies \mathcal{E}_2 . So,

$$\text{prob}\{\mathcal{E}_1\} \leq \text{Prob}\{\mathcal{E}_2\} \leq e^{-\epsilon' n}$$

by Lemma 2.

□

NOTE: According to Lemma 2, any $\alpha \geq 1$ and $c = \frac{\alpha + \log_e 4}{\epsilon_1 \epsilon_2}$ satisfy the theorem. Replacing ϵ_1, ϵ_2 with the expressions found, we get

$$c \geq 2k \left[\frac{\alpha + \log_e 4}{\epsilon' \cdot \left(1 - \min\left(\epsilon + \frac{1}{2}\epsilon', \frac{3}{4}\epsilon'\right)\right)} \right]$$

5. k-blocks of dense random graphs.

This section considers edge density $p \geq c \frac{\log n}{n}$.

THEOREM 2. For any constant integer $k > 0$ and any n and $m < \frac{n}{2k}$ there are constants $c(k), d(k) > 0$ such that the cardinality X of the biggest k -block of the graph $G_{n,p}$ with $p \geq c(k) \frac{\log n}{n}$ satisfies the property

$$\text{Prob}\{X = n-m\} \leq n^{-md(k)}$$

PROOF: Let G be an instance of $G_{n,p}$ and let the event $X = n-m$ be true in that instance. Let A be a k -block with $|A| = X$. For every $u \in V-A$, we have that

$$\left| \{u,v\} \in E(G) : v \in A \right| \leq k-1$$

(since, otherwise u would belong to A). Let

$$A_1 = \{v \in A : \exists u \in V-A : \{u,v\} \in E(G)\}$$

then

$$|A_1| \leq (k-1) |V-A| = (k-1)m.$$

Let $A_2 = A - A_1$. We get

$$|A_2| \geq n-m - (k-1)m = n-km$$

Furthermore, there is no edge from $V-A$ to A_2 .

Let \mathcal{E} be the above event. The probability of \mathcal{E} is bounded above by

$$u(m,n) = \binom{n}{m} \binom{n-m}{n-km} (1-p)^{(n-km)m} \quad (***)$$

But

$$(1-p) \leq \left(1 - \frac{c \log n}{n}\right) \leq e^{-c \frac{\log n}{n}}$$

since

$$p \geq \frac{c \log n}{n}$$

Also

$$\binom{n-m}{n-km} \leq \binom{n-m}{(k-1)m} \leq e^{(k-1)m \log(n-m)}$$

since

$$(k-1)m < \frac{n-m}{2}$$

and

$$\binom{n}{m} \leq e^{m \log n}$$

since

$$m < \frac{n}{2}$$

Thus $u(n,m) \leq n^{-d(n,m)}$ where $d(n,m) =$

$$\begin{aligned} & cm \left(1 - \frac{km}{n}\right) - m - (k-1)m \frac{\log(n-m)}{\log n} \\ & > cm \left(1 - \frac{km}{n}\right) - m - (k-1)m \\ & > \frac{c}{2}m - km \quad (\text{by our assumption}). \end{aligned}$$

So, $d(n,m) > md(k)$ where $d(k) = \frac{c}{2} - k$. Note that $d(k) > 0$ iff $c(k) > 2k$.

So, $\text{Prob}(\mathcal{E}) \leq n^{-m d(k)}$.

□

THEOREM 3: For any constant integer $k > 0$ and any $n \gg k$ there is a constant $c(k) > 0$ and a $d(k) > 0$ such that the cardinality X of the biggest k -block of the graph $G_{n,p}$ with $p \geq c(k) \frac{\log n}{n}$ satisfies the property

$$\text{Prob}\{X \leq n - \log n\} < 2n^{(1-d(k))\log n}$$

PROOF: By using theorem 2, we get

$$\text{Prob}\left\{\log n \leq n - X < \frac{n}{2k}\right\} = \sum_{m=\log n}^{n/2k} n^{-md(k)}$$

with
$$d(k) = \frac{c(k)}{2} - k > 0 \text{ for } c(k) > 2k.$$

So,
$$\text{Prob}\left\{\log n \leq n - X < \frac{n}{2k}\right\} < n \cdot n^{-\log n \cdot d(k)} < n^{1-d(k)\log n}.$$

Also, by theorem 1 and using $\epsilon = \frac{1}{2k}$ we get

$$\text{Prob}\left\{n - X > \frac{n}{2k}\right\} < e^{-\alpha \cdot n}$$

for any $\alpha > 1$ and $c(k) \geq \frac{\alpha + \log_e 4}{\epsilon_1 \epsilon_2}$ and $\epsilon_1 \epsilon_2 = \frac{1}{2k} \left(1 - \frac{3}{8k}\right)$

So, for
$$c(k) > \max\left(2k, \frac{\alpha + \log_e 4}{\epsilon_1 \epsilon_2}\right)$$

or
$$c(k) > (\alpha + \log_e 4)16k^2$$

we get

$$\text{Prob}\{\log n \leq n - X\} < e^{-\alpha \cdot n} + n^{1-\log n \cdot d(k)}$$

or

$$\text{Prob}\{X \leq n - \log n\} < 2n^{1-d(k) \cdot \log n}$$

for sufficiently large n .

□

NOTE: Theorem 3 says that for $p \geq c(k) \frac{\log n}{n}$ the graph $G_{n,p}$ has a k -block of size $\geq n - \log n$ with probability limiting to 1 as $n \rightarrow \infty$.

THEOREM 4: For any constant integer $k > 0$ and $n \gg k$ there are constants $c(k) > 0$, $d'(k) > 1$ such that the random graph $G_{n,p}$ with $p \geq c(k) \frac{\log n}{n}$ is k -connected with probability

$$\geq 1 - 2n^{-d'(k)}.$$

PROOF: Let $R = n - X$ where X = cardinality of the biggest k -block of $G_{n,p}$. By using theorems 2, 3 and $c(k) > 2 + \max\left(2k, \frac{\alpha + \log_e 4}{\epsilon_1 \epsilon_2}\right)$ with $\epsilon_1 \epsilon_2 = \frac{1}{2k} \left(1 - \frac{3}{8k}\right)$ we get that

$$\text{Prob}\{1 \leq R\} < e^{-\alpha \cdot n} + n^{\frac{1 - (\frac{c}{2} - k)}{2}}.$$

Let
$$d'(k) = \frac{c(k)}{2} - k - 1.$$

Then $d'(k) > 1$ for $c(k) > 2 + \left(\max 2k, \frac{\alpha + \log_e 4}{\epsilon_1 \epsilon_2}\right)$

and

$$\text{Prob}\{1 \leq R\} \leq e^{-\alpha \cdot n} + n^{-d'(k)} < 2n^{-d'(k)}$$

for large n .

Hence

$$\text{Prob}\{R = 0\} > 1 - 2n^{-d'(k)}$$

□

6. k-blocks for intermediate edge densities.

Let $\frac{c}{n} \leq p \leq c' \frac{\log n}{n}$. We wish to study the k-connectivity of this class of random graphs.

THEOREM 5. For any constant $k \geq 0$ and any $m = o(n)$ there is a constant $c_1(k) > 0$ and a function $t(n) > \frac{c_1(k) \log n}{m}$ such that, if $p > \frac{t(n)}{n}$ then if X is the cardinality of the biggest k-block of $G_{n,p}$ then

$$\text{Prob}\{X \leq n - m\} \leq \frac{n^k}{e^{t(n)m}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

PROOF: Assume that in the instance G of $G_{n,p}$ the cardinality X of the biggest k-block satisfies the inequality $X \leq n - m$. Then, we can find two sets Y, Z (as in proof of theorem 3) such that $|Y| = m$, $|Z| = n - km$ and no edge between them. This event is above bounded by the probability $1 - q$ where

$$q = \text{Prob}\{\text{for every pair of disjoint sets } Y, Z \text{ of vertices of the above sizes, there is at least one edge between } Y, Z.\}$$

We shall show $q \rightarrow 1$ as $n \rightarrow \infty$. Let us enumerate all possible pairs of sets of vertices of the above sizes. Call them

$$(Y_1, Z_1), (Y_2, Z_2), \dots, (Y_g, Z_g)$$

where

$$g = \binom{n}{m} \binom{n-m}{n-km} = \binom{n}{m} \binom{n-m}{(k-1)m}$$

We have that $q =$

$$\text{Prob}\{E(Y_1, Z_1) \neq \emptyset \wedge \dots \wedge E(Y_g, Z_g) \neq \emptyset\}$$

where $E(Y, Z)$ = set of edges between Y, Z .

So, by Baye's formula, $q =$

$$\text{Prob}\{E(Y_1, Z_1) \neq \emptyset\} \text{Prob}\left\{\frac{E(Y_2, Z_2) \neq \emptyset}{E(Y_1, Z_1) \neq \emptyset}\right\} \dots \text{Prob}\left\{\frac{E(Y_g, Z_g) \neq \emptyset}{\bigwedge_{i=1, \dots, g-1} E(Y_i, Z_i) \neq \emptyset}\right\}$$

We need the following enumeration lemma:

LEMMA 3: For every two sets Y_i, Z_i having at least one edge e between them, there are at least

$$g_i = \binom{n-2}{m-1} \binom{n-2-(m-1)}{(k-1)m-1}$$

pairs of sets of sizes $m, n - km$ which also have this edge between them.

This lemma can be proved easily by taking out the two vertices of e and enumerating.

COROLLARY: There is a suitable enumeration of the sets in the q product such that for every term i not equal to 1 the next g_i or more terms (conditioned on the existence of an edge from A_i to B_i) will be equal to 1.

Hence, the value of q is

$$q \geq \left[\text{Prob}\{E(Y_1, Z_1) \neq \emptyset\} \right]^{g/g_1}$$

But

$$g/g_1 \leq \left(\frac{n}{m}\right)^k \text{ as } n \rightarrow \infty.$$

Hence,

$$\begin{aligned} q &\geq \left[1 - (1-p)^{m(n-km)} \right]^{(n/m)^k} \\ &\geq \left[1 - \left((1-p)^{1/F} \right)^{pm(n-km)} \right]^{(n/m)^k} \end{aligned}$$

or

$$\begin{aligned} q &\geq \left(1 - e^{-pm(n-km)} \right)^{\left(\frac{n}{m} \right)^k} \\ &\geq 1 - \left(\frac{n}{m} \right)^k e^{-t(n)m} \end{aligned}$$

or

$$q \geq 1 - e^{-[t(n)m - k \log n]} > 1 - n^{-2}$$

if

$$c_1(k) > k + 2.$$

(Since

$$t(n)m > c_1(k) \log n > (k+2) \log n)$$

So,

$$\text{Prob}\{X < n - m\} < e^{-[t(n)m - k \log n]} \rightarrow 0 \text{ as } n \rightarrow \infty$$

for the above values of $c(k)$

COROLLARY: For $m = \log n$ and $t(n) \geq c_1(k) > k + 2$ we get: For each $k > 0$, the graph $G_{n,p}$ with $p \geq \frac{c_1(k)}{n}$, has a k -block of cardinality $> n - \log n$ with probability $\geq 1 - n^{-2}$.

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